Simple Harmonic Motion

In the previous chapter we found several new classes of equations of motion. We now add one last paradigm to our repertoire — one so powerful and ubiquitous in Physics that it deserves a chapter all to itself.

13.1 Periodic Behaviour

Nature shows us many "systems" which return periodically to the same initial state, passing through the same sequence of intermediate states every period. Life is so full of periodic experiences, from night and day to the rise and fall of the tides to the phases of the moon to the annual cycle of the seasons, that we all come well equipped with "common sense" tailored to this paradigm.[1] It has even been suggested that the concept of *time* itself is rooted in the *cyclic* phenomena of Nature.

In Physics, of course, we insist on narrowing the definition just enough to allow precision. For instance, many phenomena are *cyclic* without being *periodic* in the strict sense of the word.[2] Here *cyclic* means that the same general pattern keeps repeating; *periodic* means that the system passes through the same "phase" at *exactly* the same time in every cycle and that all the cycles are *exactly* the same length. Thus if we know all the details of *one full cycle* of true periodic behaviour, then we know the subsequent state of the system at *all* times, future and past. Naturally, this is an idealization; but its utility is obvious.

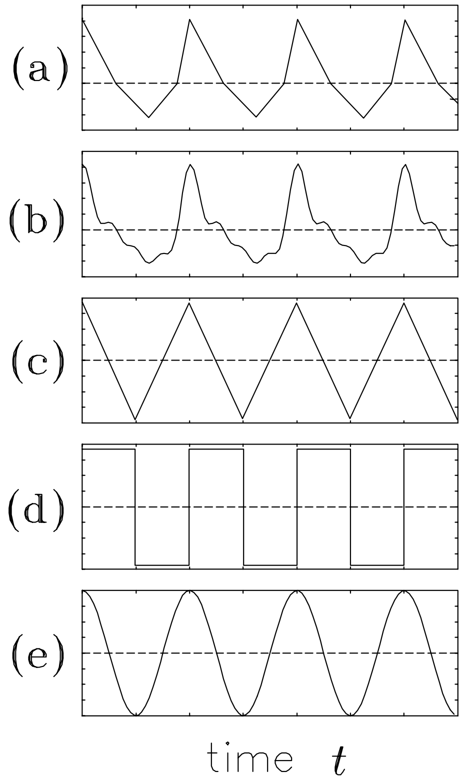

Figure 13.1 Some periodic functions.

Of course, there is an infinite variety of possible *periodic* cycles. Assuming that we can reduce the "state" of the system to a single variable "q" and its time derivatives, the graph of $q(t)$

[1] Many people are so taken with this paradigm that they apply it to all experience. The *I Ching*, for instance, is said to be based on the ancient equivalent of "tuning in" to the "vibrations" of Life and the World so that one's awareness resonates with the universe. By New Age reckoning, cultivating such resonances is supposed to be the fast track to enlightenment. Actually, Physics relies very heavily on the same paradigm and in fact supports the notion that many apparently random phenomena are actually superpositions of regular cycles; however, it offers little encouragement for expecting "answers" to emerge effortlessly from such a tuning of one's mind's resonances. Too bad. But I'm getting ahead of myself here.

[2] Examples of *cyclic* but not necessarily *periodic* phenomena are the mass extinctions of species on Earth that seem to have occurred roughly every 24 million years, the "seven-year cycle" of sunspot activity, the return of salmon to the river of their origin and recurring droughts in Africa. In some cases the basic reason for the cycle is understood and it is obvious why it only repeats approximately; in other cases we have no idea of the root cause; and in still others there is not even a consensus that the phenomenon is truly cyclic — as opposed to just a random fluctuation that just happens to mimic cyclic behaviour over a short time. Obviously the resolution of these uncertainties demands "more data," *i.e.* watching to see if the cycle continues; with the mass extinction "cycle," this requires considerable patience. When "periodicity debates" rage on in the absence of additional data, it is usually a sign that the combatants have some other axe to grind.

can have any shape as long as it *repeats* after one full period. Fig. 13.1 illustrates a few examples. In (a) and (b) the "displacement" of q away from its "equilibrium" position [dashed line] is not symmetric, yet the phases repeat every cycle. In (c) and (d) the cycle is symmetric with the same "amplitude" above and below the equilibrium axis, but at certain points the slope of the curve changes "discontinuously." Only in (e) is the cycle everywhere smooth and symmetric.

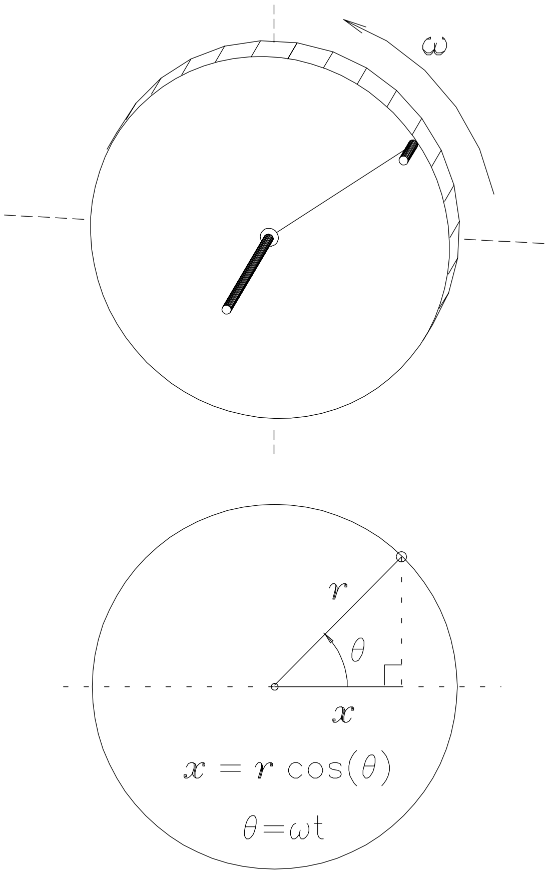

Figure 13.2 Projected motion of a point on the rim of a wheel.

13.2 Sinusoidal Motion

There is one sort of periodic behaviour that is mathematically the *simplest possible* kind. This is the "sinusoidal" motion shown in Fig. 13.1(e), so called because one realization is the sine function, $\sin(x)$. It is easiest to see this by means of a crude mechanical example.

13.2.1 Projecting the Wheel

Imagine a rigid wheel rotating at constant angular velocity about a fixed central axle. A bolt screwed into the rim of the wheel executes *uniform circular motion* about the centre of the axle.[3] For reference we scribe a line on the wheel from the centre straight out to the bolt and call this line the *radius vector*. Imagine now that we take this apparatus outside at high noon and watch the motion of the *shadow* of the bolt on the ground. This is (naturally enough) called the *projection* of the circular motion onto the horizontal axis. At some instant the radius vector makes an angle $\theta = \omega t + \phi$ with the horizontal, where ω is the angular frequency of the wheel (2π times the number of full revolutions per unit time) and ϕ is the initial angle (at $t = 0$) between the radius vector and the horizontal.[4] From a side view of the wheel we can see that the distance x from the shadow of the central axle to the shadow of the bolt [*i.e.* the *projected horizontal displacement* of the bolt from the centre, where $x = 0$] will be given by trigonometry on the indicated right-angle triangle:

$$\cos(\theta) \equiv \frac{x}{r} \quad \Rightarrow \quad x = r\cos(\theta) = r\cos(\omega t + \phi) \tag{1}$$

The resultant *amplitude* of the displacement as a *function of time* is shown in Fig. 13.3.

The *horizontal velocity* v_x of the projected shadow of the bolt on the ground can also be obtained by trigonometry if we recall that the

[3]Note the frequency with which we periodically recycle our paradigms!

[4]The inclusion of the "initial phase" ϕ makes this description completely general.

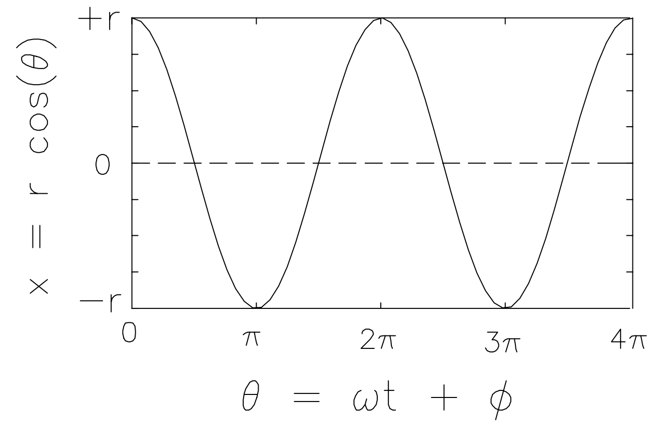

Figure 13.3 The cosine function.

vector velocity $\vec{\boldsymbol{v}}$ is always perpendicular to the radius vector $\vec{\boldsymbol{r}}$. I will leave it as an exercise for the reader to show that the result is

$$v_x \;=\; -\,v \,\sin(\theta) \;=\; -\,r\,\omega\,\sin(\omega t + \phi) \qquad (2)$$

where $v = r\omega$ is the constant speed of the bolt in its circular motion around the axle. It also follows (by the same sorts of arguments) that the *horizontal acceleration* a_x of the bolt's shadow is the projection onto the $\hat{\boldsymbol{x}}$ direction of $\vec{\boldsymbol{a}}$, which we know is back toward the centre of the wheel — *i.e.* in the $-\hat{\boldsymbol{x}}$ direction; its value at time t is given by

$$a_x \;=\; -\,a\,\cos(\theta) \;=\; -\,r\,\omega^2\,\cos(\omega t + \phi) \qquad (3)$$

where $a = \dfrac{v^2}{r} = r\omega^2$ is the magnitude of the centripetal acceleration of the bolt.

13.3 Simple Harmonic Motion

The above mechanical example serves to introduce the idea of $\cos(\theta)$ and $\sin(\theta)$ as *functions* in the sense to which we have (I hope) now become accustomed. In particular, if we realize that (by definition) $v_x \equiv \dot{x}$ and $a_x \equiv \ddot{x}$, the formulae for $v_x(t)$ and $a_x(t)$ represent the *derivatives* of $x(t)$:

$$\begin{aligned} x &= r\,\cos(\omega t + \phi) && (4)\\ \dot{x} &= -\,r\,\omega\,\sin(\omega t + \phi) && (5)\\ \ddot{x} &= -\,r\,\omega^2\,\cos(\omega t + \phi) && (6) \end{aligned}$$

— which in turn tell us the derivatives of the sine and cosine functions:

$$\begin{aligned} \frac{d}{dt}\,\cos(\omega t + \phi) &= -\omega\,\sin(\omega t + \phi) && (7)\\ \frac{d}{dt}\,\sin(\omega t + \phi) &= \omega\,\cos(\omega t + \phi) && (8) \end{aligned}$$

So if we want we can calculate the n^{th} derivative of a sine or cosine function almost as easily as we did for our "old" friend the exponential function. I will not go through the details this time, but this feature again allows us to express these functions as *series expansions*:

$$\begin{array}{lllllll} \exp(z) = & 1 & +z & +\frac{1}{2}z^2 & +\frac{1}{3!}z^3 & +\frac{1}{4!}z^4 & +\cdots \\ \cos(z) = & 1 & & -\frac{1}{2}z^2 & & +\frac{1}{4!}z^4 & -\cdots \\ \sin(z) = & & z & & -\frac{1}{3!}z^3 & & +\cdots \end{array} \qquad (9)$$

where I have shown the exponential function along with the sine and cosine for reasons that will soon be apparent.

It is definitely worth remembering the SMALL ANGLE APPROXIMATIONS

$$\begin{array}{rl} \text{For} \quad \theta \ll 1, & \cos(\theta) \;\approx\; 1 \;-\; \frac{1}{2}\theta^2 \\ \text{and} & \sin(\theta) \;\approx\; \theta. \end{array} \qquad (10)$$

13.3.1 The Spring Pendulum

Another mecahnical example will serve to establish the paradigm of SIMPLE HARMONIC MOTION (*SHM*) as a solution to a particular type of *equation of motion*.[5]

[5] Although we have become conditioned to expect such *mathematical* formulations of relationships to be more removed from our intuitive understanding than easily visualized *concrete* examples like the projection of circular motion, this

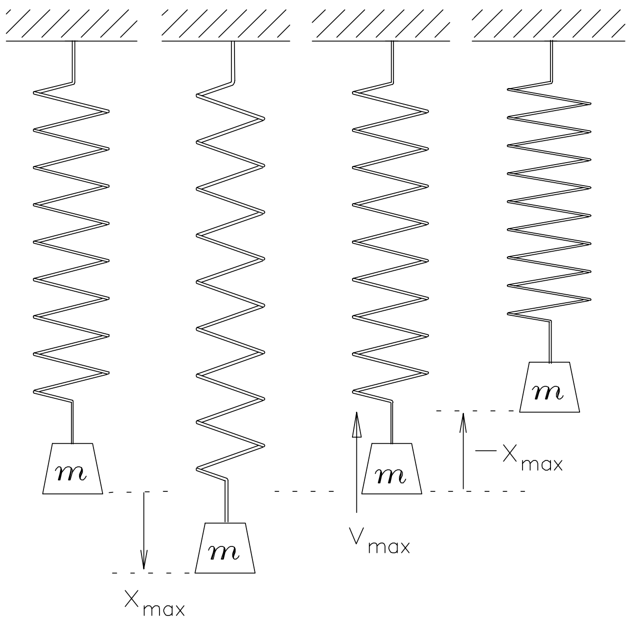

Figure 13.4 Successive "snapshots" of a mass bouncing up and down on a spring.

As discussed in a previous chapter, the *spring* embodies one of Physics' premiere paradigms, the *linear restoring force*. That is, a force which disappears when the system in question is in its "equilibrium position" x_0 [which we will define as the $x = 0$ position ($x_0 \equiv 0$) to make the calculations easier] but increases as x moves away from equilibrium, in such a way that the *magnitude* of the force F is proportional to the displacement from equilibrium [F is *linear* in x] and the *direction* of F is such as to try to *restore* x to the original position. The constant of proportionality is called the *spring constant*, always written k. Thus $F = -kx$ and the resultant equation of motion is

$$\ddot{x} = -\left(\frac{k}{m}\right) x \tag{11}$$

Note that the *mass* plays a rôle just as essential as the *linear restoring force* in this paradigm. If $m \to 0$ in this equation, then the acceleration becomes infinite and in principle the spring would just return instantaneously to its equilibrium length and stay there!

In the leftmost frame of Fig. 13.4 the mass m is at rest and the spring is in its equilibrium position (*i.e.* neither stretched nor compressed) defined as $x = 0$. In the second frame, the spring has been gradually pulled down a distance $x_{\max}$ and the mass is once again at rest. Then the mass is released and accelerates upward under the influence of the spring until it reaches the equilibrium position again [third frame]. This time, however, it is moving at its maximum velocity $v_{\max}$ as it crosses the centre position; as soon as it goes higher, it *compresses* the spring and begins to be *decelerated* by a linear restoring force in the opposite direction. Eventually, when $x = -x_{\max}$, all the kinetic energy has been been stored back up in the compression of the spring and the mass is once again instantaneously at rest [fourth frame]. It immediately starts moving downward again at maximum acceleration and heads back toward its starting point. In the absence of friction, this cycle will repeat forever.

I now want to call your attention to the acute similarity between the above differential equation and the one we solved for exponential decay:

$$\dot{x} = -\kappa \, x \tag{12}$$

and, by extension,

$$\ddot{x} = \kappa^2 \, x \tag{13}$$

the solution to which equation of motion (*i.e.* the function which "satisfies" the differential equation) was

$$x(t) = x_0 \, e^{-\kappa \, t} \tag{14}$$

Now, if only we could equate κ^2 with $-k/m$, these equations of motion (and therefore their

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is a case where the mathematics allows us to draw a sweeping conclusion about the detailed behaviour of *any* system that exhibits certain simple properties. Furthermore, these conditions *are actually satisfied* by an incredible variety of *real* systems, from the atoms that make up any solid object to the interpersonal "distance" in an intimate relationship. Just wait!

solutions) would be exactly the same! The problem is, of course, that both k and m are intrinsically positive constants, so it is tough to find a real constant κ which gives a negative number when squared.

Imaginary Exponents

Mathematics, of course, provides a simple solution to this problem: just have κ be an *imaginary* number, say

$$\kappa \equiv i\,\omega \qquad \text{where} \qquad i \equiv \sqrt{-1}$$

and ω is a positive real constant. Let's see if this trial solution "works" (*i.e.* take its second derivative and see if we get back our equation of motion):

$$\begin{aligned} x(t) &= x_0\, e^{i\,\omega\, t} && (15)\\ \dot{x} &= i\,\omega\, x_0\, e^{i\,\omega\, t} && (16)\\ \ddot{x} &= -\omega^2\, x_0\, e^{i\,\omega\, t} && (17)\\ \text{or} \qquad \ddot{x} &= -\omega^2\, x && (18)\\ \text{so} \qquad \omega &\equiv \sqrt{\frac{k}{m}} && (19) \end{aligned}$$

OK, it works. But what does it *describe*? For this we go back to our series expansions for the exponential, sine and cosine functions and note that *if we let* $z \equiv i\theta$, the following *mathematical identity* holds:[6]

$$e^{i\,\theta} = \cos(\theta) + i\, \sin(\theta) \qquad (20)$$

Thus, for the case at hand, if $\theta \equiv \omega\, t$ [you probably knew this was coming] then

$$x_0\, e^{i\,\omega\, t} = x_0\, \cos(\omega t) + i\, x_0\, \sin(\omega t)$$

— *i.e.* the formula for the projection of uniform circular motion, with an imaginary part "tacked on." (I have set the initial phase ϕ to zero just to keep things simple.) What does *this* mean?

I don't know.

What! How can I say, "I don't know," about the premiere paradigm of Mechanics? We're supposed to know *everything* about Mechanics! Let me put it this way: we have happened upon a nice tidy mathematical representation that *works* — *i.e.* if we use certain rules to manipulate the mathematics, it will faithfully give correct answers to our questions about how this thing will behave. The rules, by the way, are as follows:

> Keep the imaginary components through all your calculations until the final "answer," and then *throw away* any remaining *imaginary parts* of any actual *measurable quantity*.

The point is, there is a difference between understanding how something *works* and knowing what it *means*. Meaning is something we put into our world by act of will, though not always conscious will. How it works is there before us and after we are gone. No one asks the "meaning" of a screwdriver or a carburetor or a copy machine; some of the conceptual tools of Physics are in this class, though of course there is nothing to prevent anyone from *putting* meaning into them.[7]

13.4 Damped Harmonic Motion

Let's take stock. In the previous chapter we found that

$$x(t) = [\text{constant}] - \frac{v_0}{\kappa}\, e^{-\kappa\, t}$$

[6] You may find this unremarkable, but I have never gotten over my astonishment that functions so ostensibly unrelated as the *exponential* and the *sinusoidal* functions could be so intimately connected! And for once the mathematical oddity has profound *practical applications*.

[7] I am reminded of a passage in one of Kurt Vonnegut's books, perhaps *Sirens of Titan*, in which the story of creation is told something like this: God creates the world; then he creates Man, who sits up, looks around and says, "What's the meaning of all this?" God answers, "What, there has to be a meaning?" Man: "Of course." God: "Well then, I leave it to you to think of one."

satisfies the basic differential equation

$$\ddot{x} = -\kappa \dot{x} \qquad \text{or} \qquad a = -\kappa v$$

defining *damped* motion (*e.g.* motion under the influence of a frictional force proportional to the velocity). We now have a solution to the equation of motion defining *SHM*,

$$\ddot{x} = -\omega^2 x \quad \Rightarrow \quad x(t) = x_0 \, e^{i\,\omega\,t},$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

and I have set the initial phase ϕ to zero just to keep things simple. Can we put these together to "solve" the more general (and realistic) problem of *damped harmonic motion*? The differential equation would then read

$$\ddot{x} \;=\; -\omega^2 x \;-\kappa \, \dot{x} \tag{21}$$

which is beginning to look a little hard. Still, we can sort it out: the first term on the *RHS* says that there is a linear restoring force and an inertial factor. The second term says that there is a damping force proportional to the velocity. So the differential equation itself is not all that fearsome. How can we "solve" it?

As always, by trial and error. Since we have found the *exponential* function to be so useful, let's try one here: *Suppose* that

$$x(t) = x_0 \, e^{K\,t} \tag{22}$$

where x_0 and K are unspecified constants. Now plug this into the differential equation and see what we get:

$$\dot{x} \;=\; K \, x_0 \, e^{K\,t} \;=\; K \, x$$

and

$$\ddot{x} \;=\; K^2 \, x_0 \, e^{K\,t} \;=\; K^2 \, x$$

The whole thing then reads

$$K^2 \, x \;=\; -\omega^2 \, x \;-\kappa \, K \, x$$

which can be true "for all x" only if

$$K^2 = -\omega^2 \;- \kappa \, K \qquad \text{or} \qquad K^2 + \kappa K + \omega^2 = 0$$

which is in the standard form of a general quadratic equation for K, to which there are two solutions:

$$K = \frac{-\kappa \pm \sqrt{\kappa^2 - 4\omega^2}}{2} \tag{23}$$

Either of the two solutions given by substituting Eq. (23) into Eq. (22) will satisfy Eq. (21) describing *damped SHM*. In fact, generally any *linear combination* of the two solutions will also be a solution. This can get complicated, but we *have* found the answer to a rather broad question.

13.4.1 Limiting Cases

Let's consider a couple of "limiting cases" of such solutions. First, suppose that the linear restoring force is extremely weak compared to the "drag" force — *i.e.*[8] $\kappa \gg \omega = \sqrt{\frac{k}{m}}$. Then $\sqrt{\kappa^2 - 4\omega^2} \approx \kappa$ and the solutions are $K \approx 0$ [*i.e.* $x \approx$ constant, plausible only if $x = 0$] and $K \approx -\kappa$, which gives the same sort of damped behaviour as if there were no restoring force, which is what we expected.

Now consider the case where the linear restoring force is very strong and the "drag" force extremely weak — *i.e.* $\kappa \ll \omega = \sqrt{\frac{k}{m}}$. Then $\sqrt{\kappa^2 - 4\omega^2} \approx 2\,i\,\omega$ and the solutions are $K \approx -\frac{1}{2}\kappa \pm i\,\omega$, or[9]

$$\begin{aligned} x(t) &= x_0 \; e^{K} && (24)\\ &\approx x_0 \; \exp(\pm i\,\omega t \;- \gamma t) && (25)\\ &= x_0 \; e^{\pm i\,\omega t} \cdot e^{-\gamma t} && (26) \end{aligned}$$

[8] The "$\gg$" symbol means "...is *much* greater than..." — there is an analogous "$\ll$" symbol that means "...is much *less* than...."

[9] There is a general rule about *exponents* that says, "A number raised to the sum of two powers is equal to the product of the same number raised to each power separately," or

$$a^{b+c} = a^b \cdot a^c.$$

where $\gamma \equiv \frac{1}{2}\kappa$. We may then think of $[i\,K]$ as a *complex frequency*[10] whose real part is $\pm\omega$ and whose imaginary part is γ. What sort of situation does this describe? It describes a *weakly damped harmonic motion* in which the usual sinusoidal pattern damps away within an "envelope" whose shape is that of an exponential decay. A typical case is shown in Fig. 13.5.

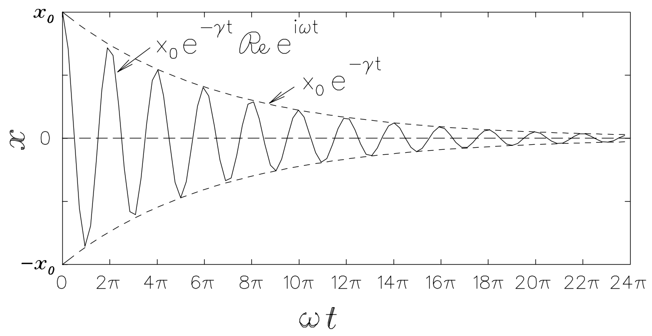

Figure 13.5 Weakly damped harmonic motion. The initial amplitude of x (whatever x is) is x_0, the angular frequency is ω and the damping rate is γ. The cosine-like oscillation, equivalent to the *real* part of $x_0\, e^{i\,\omega\, t}$, decays within the *envelope function* $x_0\, e^{-\gamma\, t}$.

13.5 Generalization of $\mathcal{SHM}$

As for all the other types of *equations of motion*, *SHM* need not have anything to do with masses, springs or even Physics. Even within Physics, however, there are so many different kinds of examples of *SHM* that we go out of our way to generalize the results: using "q" to represent the "coordinate" whose displacement from the equilibrium "position" (always taken as $q = 0$) engenders some sort of restoring "force" $Q = -k\,q$ and "μ" to represent an "inertial factor" that plays the rôle of the mass, we have

$$\ddot{q} = -\left(\frac{k}{\mu}\right) q \qquad (27)$$

for which the solution is the real part of

$$q(t) = q_0\, e^{i\,\omega\, t} \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{\mu}} \qquad (28)$$

When some form of "drag" acts on the system, we expect to see the qualitative behaviour pictured in Fig. 13.5 and described by Eqs. (22) and (23). Although one might expect virtually every real example to have some sort of frictional damping term, in fact there are numerous physical examples with no damping whatsoever, mostly from the microscopic world of solids.

13.6 The Universality of $\mathcal{SHM}$

If two systems satisfy the same equation of motion, their behaviour is the same. Therefore the motion of the mass on the spring is *in every respect identical* to the horizontal component of the motion of the peg in the rotating wheel, even though these two systems are physically quite distinct. In fact, *any* system exhibiting both a LINEAR RESTORING "FORCE" and an INERTIAL FACTOR analogous to MASS will exhibit *SHM*.[11] Moreover, since these arguments may be used equally well in reverse, the horizontal component of the *force* acting on the peg in the wheel must obey $F_x = -kx$, where k is an "effective spring constant."

[10] The word "complex" has, like "real" and "imaginary," been ripped off by Mathematicians and given a very explicit meaning that is not entirely compatible with its ordinary dictionary definition. While a *complex* number in Mathematics may indeed be complex — *i.e.* complicated and difficult to understand — it is *defined* only by virtue of its having both a *real* part and an *imaginary* part, such as $z = a + i\,b$, where a and b are both *real*. I hope that makes everything crystal clear. . . .

[11] Examples are plentiful, especially in view of the fact that *any* potential energy minimum is approximately quadratic for small enough displacements from equilibrium. A prime example from outside Mechanics is the *electrical circuit*, in which the charge Q on a capacitor plays the rôle of the displacement variable x and the inertial factor is provided by an inductance, which resists changes in the current $I = dQ/dt$.

Why is *SHM* characteristic of such an enormous variety of phenomena? Because *for sufficiently small* displacements from equilibrium, *every* system with an equilibrium configuration satisfies the first condition for *SHM*: the linear restoring force. Here is the simple argument: a linear restoring force is equivalent to a potential energy of the form $U(q) = \frac{1}{2}k\,q^2$ — *i.e.* a "quadratic minimum" of the potential energy at the equilibrium configuration $q = 0$. But if we "blow up" a graph of $U(q)$ near $q = 0$, *every* minimum looks quadratic under sufficient magnification! That means *any* system that *has* an equilibrium configuration also has some analogue of a "potential energy" which is a minimum there; if it also has some form of *inertia* so that it tends to stay at rest (or in motion) until acted upon by the analogue of a *force*, then it will automatically exhibit *SHM* for small-amplitude displacements. This makes *SHM* an extremely powerful paradigm.

13.6.1 Equivalent Paradigms

We have established previously that a LINEAR RESTORING FORCE $F = -kx$ is completely equivalent to a QUADRATIC MINIMUM IN POTENTIAL ENERGY $U = \frac{1}{2}kx^2$. We now find that, with the inclusion of an INERTIAL FACTOR (usually just the MASS m), either of these *physical paradigms* will guarantee the *mathematical paradigm* of *SHM* — *i.e.* the displacement x from equilibrium will satisfy the equation of motion

$$x(t) = x_{\max} \cos(\omega t + \phi) \qquad (29)$$

where $x_{\max}$ is the *amplitude* of the oscillation. Any $x(t)$ of this form automatically satisfies the definitive equation of motion of *SHM*, namely

$$\frac{d^2x}{dt^2} = -\omega^2\, x \qquad (30)$$

and *vice versa* — whenever x satisfies Eq. (30), the explicit time dependence of x will be given by Eq. (29).

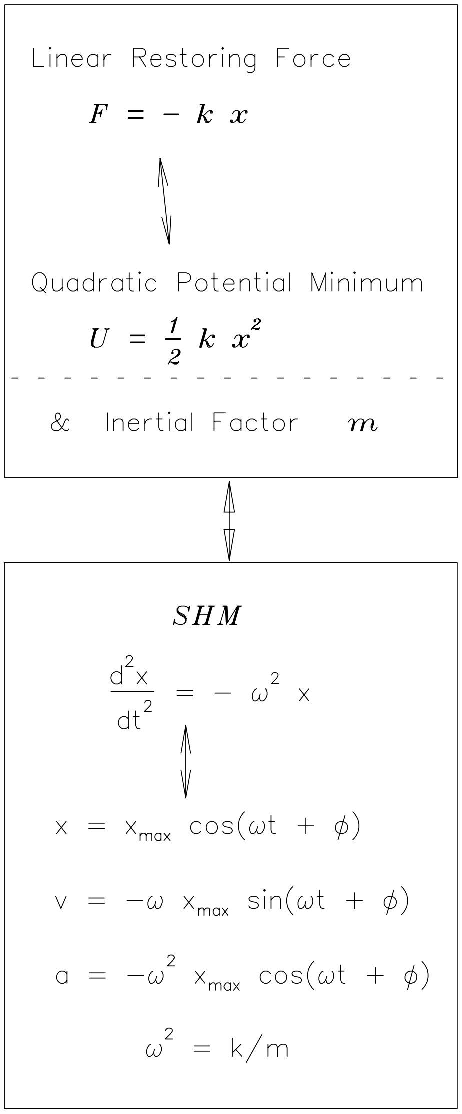

Figure 13.6 Equivalent paradigms of *SHM*.

13.7 Resonance

No description of *SHM* would be complete without some discussion of the general phenomenon of *resonance*, which has many practical consequences that often seem very counterintuitive.[12] I will, however, overcome my zeal for demonstrating the versatility of Mathematics and stick to a simple qualitative description of resonance.

[12]It is, after all, one of the main purposes of this book to dismantle your intuition and rebuild it with the faulty parts left out and some shiny new paradigms added.

Just this once.

The basic idea is like this: suppose some system exhibits all the requisite properties for *SHM*, namely a linear restoring "force" $Q = -k\,q$ and an inertial factor μ. Then *if once set in motion* it will oscillate forever at its "resonant frequency" $\omega = \sqrt{\frac{k}{\mu}}$, unless of course there is a "damping force" $D = -\kappa\mu q$ to dissipate the energy stored in the oscillation. As long as the damping is weak $[\kappa \ll \sqrt{\frac{k}{m}}]$, any oscillations will persist for many periods. Now suppose the system is initially at rest, in equilibrium, ho hum. What does it take to "get it going?"

The *hard* way is to give it a great whack to start it off with lots of kinetic energy, or a great tug to stretch the "spring" out until it has lots of potential energy, and then let nature take its course. The *easy* way is to give a tiny push to start up a small oscillation, then wait exactly one full period and give another tiny push to increase the amplitude a little, and so on. This works because *the frequency* ω *is independent of the amplitude* q_0. So if we "drive" the system *at its natural "resonant" frequency* ω, no matter how small the individual "pushes" are, we will slowly build up an *arbitrarily large oscillation.*[13]

Such resonances often have dramatic results. A vivid example is the famous movie of the collapse of the Tacoma Narrows bridge, which had a torsional [twisting] resonance[14] that was excited by a steady breeze blowing past the bridge. The engineer in charge anticipated all the other more familiar resonances [of which there are many] and incorporated devices specifically designed to safely damp their oscillations, but forgot this one. As a result, the bridge developed huge twisting oscillations [mistakes like this are usually painfully obvious when it is too late to correct them] and tore itself apart.

A less spectacular example is the trick of getting yourself going on a playground swing by leaning back and forth with arms and legs in synchrony with the natural frequency of oscillation of the swing [a sort of pendulum]. If your kinesthetic memory is good enough you may recall that it is important to have the "driving" push exactly $\frac{\pi}{2}$ radians [a quarter cycle] "out of phase" with your velocity — *i.e.* you *pull* when you reach the *motionless* position at the top of your swing, if you want to achieve the maximum result. This has an elegant mathematical explanation, but I promised. . . .

[13]Of course, this assumes $\kappa = 0$. If damping occurs at the same time, we must put at least as much energy *in* with our driving force as friction takes *out* through the damping in order to build up the amplitude. Almost every system has some limiting amplitude beyond which the restoring force is no longer linear and/or some sort of losses set in.

[14](something like you get from a blade of grass held between the thumbs to create a loud noise when you blow past it)