Special Relativity

Let's briefly recapitulate the situation in 1865: MAXWELL'S EQUATIONS, which correctly described all the phenomena of electromagnetism known in the mid-19th Century (and then some), predicted also that electromagnetic fields should satisfy the WAVE EQUATION — *i.e.*, by virtue of a changing $\vec{\boldsymbol{E}}$ creating $\vec{\boldsymbol{B}}$ and *vice versa*, the electric and magnetic fields would be able to "play off each other" and propagate through space in the form of a *wave* with all the properties of *light* (or its manifestations in shorter and longer wavelengths, which we also term "light" when discussing electromagnetic waves in general). Fine, so far.

But there are some unsettling implications of this "final" explanation of light. First of all (and the focus of this Chapter) is the omission of any reference to a *medium* that does the "wiggling" as the electromagnetic wave goes through it. Water waves propagate through water, sound waves through air, liquid or solid, plasma waves through plasmas, *etc.* This was the first time anyone had ever postulated a wave that *just propagated by itself* through *empty vacuum* (or "free space," as it is often called in this context). Moreover, the propagation velocity of light (or any electromagnetic wave) through the vacuum is given unambiguously by MAXWELL'S EQUATIONS to be $c = 2.99792458 \times 10^8$ m/s, *regardless of the motion of the observer.*

23.1 Galilean Transformations

So what? Well, this innocuous looking claim has some *very* perplexing logical consequences with regard to *relative velocities*, where we have expectations that follow, seemingly, from self-evident common sense. For instance, suppose the propagation velocity of ripples (water waves) in a calm lake is 0.5 m/s. If I am walking along a dock at 1 m/s and I toss a pebble in the lake, the guy sitting at anchor in a boat will see the ripples move by at 0.5 m/s but I will see them *dropping back* relative to me! That is, I can "outrun" the waves. In mathematical terms, if all the velocities are in the same direction (say, along x), we just *add* relative velocities: if v is the velocity of the wave relative to the water and u is my velocity relative to the water, then v', the velocity of the wave relative to *me*, is given by $v' = v - u$. This common sense equation is known as the GALILEAN VELOCITY TRANSFORMATION — a big name for a little idea, it would seem.

With a simple diagram, we can summarize the common-sense GALILEAN TRANSFORMATIONS (named after Galileo, no Biblical reference):

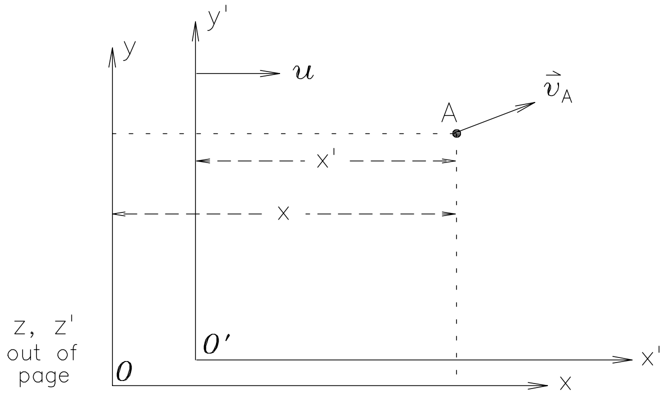

Figure 23.1 Reference frames of a "stationary" observer O and an observer O' moving in the x direction at a velocity u relative to O. The coordinates and time of an event at A measured by observer O are $\{x, y, z, t\}$ whereas the coordinates and time of the *same* event measured by O' are $\{x', y', z', t'\}$. An object at A moving at velocity $\vec{\boldsymbol{v}}_A$ relative to observer O will be moving at a different velocity $\vec{\boldsymbol{v}}'_A$ in the reference frame of O'. For convenience, we always assume that O and O' coincide initially, so that everyone agrees about the "origin:" when $t = 0$ and $t' = 0$, $x = x'$, $y = y'$ and $z = z'$.

First of all, it is self-evident that $t' = t$, otherwise nothing would make any sense at all.[1] Nevertheless, we include this explicitly. Similarly, if the relative motion of O' with respect to O is only in the x direction, then $y' = y$ and $z' = z$, which were true at $t = t' = 0$, must remain true at all later times. In fact, the only coordinates that *differ* between the two observers are x and x'. After a time t, the distance (x') from O' to some object A is *less* than the distance (x) from O to A by an amount ut, because that is how much *closer* O' has *moved* to A in the interim. Mathematically, $x' = x - ut$.

The *velocity* $\vec{\boldsymbol{v}}_A$ of A in the reference frame of O also looks different when viewed from O' — namely, we have to subtract the relative velocity of O' with respect to O, which we have labelled $\vec{\boldsymbol{u}}$. In this case we picked $\vec{\boldsymbol{u}}$ along $\hat{x}$, so that the vector subtraction $\vec{\boldsymbol{v}}'_A = \vec{\boldsymbol{v}}_A - \vec{\boldsymbol{u}}$ becomes just $v'_{A_x} = v_{A_x} - u$ while $v'_{A_y} = v_{A_y}$ and $v'_{A_z} = v_{A_z}$. Let's summarize all these "coordinate transformations:"

[1] By now, this phrase should alert you to the likelihood of error.

The GALILEAN TRANSFORMATIONS:

Coordinates:

$$x' = x - ut \quad (1)$$
$$y' = y \quad (2)$$
$$z' = z \quad (3)$$
$$t' = t \quad (4)$$

Velocities:

$$v'_{A_x} = v_{A_x} - u \quad (5)$$
$$v'_{A_y} = v_{A_y} \quad (6)$$
$$v'_{A_z} = v_{A_z} \quad (7)$$

This is all so simple and obvious that it is hard to focus one's attention on it. We take all these properties for granted — and therein lies the danger.

23.2 Lorentz Transformations

The problem is, *it doesn't work for light*. Without any *stuff* with respect to which to measure relative velocity, one person's vacuum looks exactly the same as another's, even though they may be moving past each other at enormous velocity! If so, then MAXWELL'S EQUATIONS tell *both* observers that they should "see" the light go past them at c, even though one *observer* might be moving at $\frac{1}{2}c$ relative to the other!

The only way to make such a description *self-consistent* (not to say reasonable) is to allow *length* and *duration* to be different for observers moving relative to one another. That is, x' and t' must differ from x and t *not only* by additive constants but also by a *multiplicative factor*.

For æsthetic reasons I will reproduce here the equations that provide such coordinate transformations; the derivation will come later.

The ubiquitous factor γ is equal to 1 for vanishingly small relative velocity u and grows without limit as $u \to c$. In fact, if u ever got as big as c then γ would "blow up" (become infinite) and then (worse yet) become *imaginary* for $u > c$.

The LORENTZ TRANSFORMATIONS:

Coordinates:

$$x' = \gamma\,(x - ut) \quad (8)$$
$$y' = y \quad (9)$$
$$z' = z \quad (10)$$
$$t' = \gamma\,(t - \frac{ux}{c^2}) \quad (11)$$

Velocities:

$$v'_{A_x} = \frac{v_{A_x} - u}{1 - uv_{A_x}/c^2} \quad (12)$$
$$v'_{A_y} = \frac{v_{A_y}}{\gamma\,(1 - uv_{A_x}/c^2)} \quad (13)$$
$$v'_{A_z} = \frac{v_{A_z}}{\gamma\,(1 - uv_{A_x}/c^2)} \quad (14)$$

$$\text{where} \quad \beta \equiv \frac{u}{c} \quad (15)$$
$$\text{and} \quad \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \quad (16)$$

23.3 The Luminiferous Æther

This sort of nonsense convinced most people that MAXWELL'S EQUATIONS were *wrong* — or, more charitably, *incomplete*. The obvious way out of this dilemma was to assume that what we perceive (in our ignorance) as *vacuum* is actually an extremely peculiar *substance* called the "luminiferous æther" through which ordinary "solid" matter passes more or less freely but in which the "field lines" of electromagnetism are actual "ripples." (Sort of.) This recovers the rationalizing influence of a *medium* through which light propagates, at the expense of some pretty unfamiliar properties of the medium. [You can see the severity of the dilemma in the lengths to which people were willing to go to find a way out of it.] All that remained was to find a way of measuring the *observer's* velocity *relative to the æther*.

Since "solid" objects slip more or less effortlessly through the æther, this presented some problems. What was eventually settled for was to measure the *apparent speed of light* propagation in different directions; since we are moving through the æther, the light should *appear* to propagate more slowly in the direction *we* are moving, since we are then catching up with it a little.[2]

[2]Recall the image of the pebble-thrower walking along the dock and watching the ripples propagate in the pond.

23.3.1 The Speed of Light

The speed of light is so enormous (299,792 km/s) that we scarcely notice a delay between the transmission and reception of electromagnetic waves under normal circumstances. However, the same electronic technology that raised all these issues in the first place also made it possible to perform *timing* to a precision of millionths of a second (microseconds [μs]) or even billionths of a second (nanoseconds [ns]). Today we routinely send telephone signals out to geosynchronous satellites and back (a round trip of at least 70,800 km) with the result that we often notice [and are irritated by] the delay of 0.236 seconds or more in transoceanic telephone conversations. For computer communications this delay is even more annoying, which was a strong motive for recently laying optical fiber communications cables under the Atlantic and Pacific oceans! So we are already bumping up against the limitations of the finite speed of light in our "everyday lives" (well, almost) without any involvement of the weird effects in this Chapter!

23.3.2 Michelson-Morley Experiment

The famous experiment of Albert Abraham Michelson and Edward Williams Morley actually involved an *interferometer* — a device that measures how much *out of phase* two waves get when one travels a certain distance North and South while the other travels a different distance East and West. Since one of these signals may have to "swim upstream" and then downstream against the æther flowing past the Earth, it will lose a little ground overall relative to the one that just goes "across" and back, with the result that it gets out of phase by a wavelength or two. There is no need to know the *exact* phase difference, because one can simply *rotate the interferometer* and watch as one gets behind the other and then *vice versa*. When Michelson and Morley first used this ingenious device to measure the velocity of the Earth through the æther, they got an astonishing result: *the Earth was at rest!*

Did Michelson or Morley experience brief paranoid fantasies that the ergocentric doctrines of the Mediæval Church might have been right after all? Probably not, but we shall never know. Certainly they assumed they had made some mistake, since their result implied that the Earth was, at least at that moment, at rest with respect to the Universe-spanning luminiferous æther, and hence in some real sense at the centre of the Universe. However, repeating the measurement gave the same result.

Fortunately, they knew they had only to wait six months to try again, since at that time the Earth would be on the opposite side of the Sun, moving in the opposite direction relative to it (the Sun) at its orbital velocity, which should be easily detected by their apparatus. This they did, and obtained the same result. The Earth was *still* at rest relative to the æther.

Now everyone was in a bind. If they insisted in positing an æther to dispell the absurdities of propagation through a vacuum at a fixed velocity, then they had to adopt the embarrassing view that the æther actually chose the Earth, of all the heavenly bodies, to define its rest frame — *and even followed it around* in its accelerated orbital path! This was too much.

23.3.3 FitzGerald/Lorentz Æther Drag

George Francis FitzGerald and H.A. Lorentz offered a solution of sorts: in drifting through the æther, "solid" bodies were not perfectly unaffected by it but in fact suffered a common "drag" in the direction of motion that caused all the yardsticks to be "squashed" in that direction, so that the apparatus *seemed* to be unaffected only because the apparatus and the yardstick and the experimenters' eyeballs were all *contracted* by exactly the same multiplicative factor! They showed by simple arguments that said factor was in fact $\gamma = 1/\sqrt{1-\beta^2}$ where $\beta = u/c$ — *i.e.* exactly the factor defined earlier in the LORENTZ TRANSFORMATIONS, so named after one of their originators![3] Their equations were right, but their explanation (though no more outlandish than what we now believe to be correct) was wrong.

For one thing, these famous "LORENTZ CONTRACTIONS" of the lengths of meter or yardsticks were not accompanied (in their model) by any change in the relative lengths of *time* intervals — how could they be? Such an idea makes no sense! But this leads to qualitative inconsistencies in the descriptions of sequences of events as described by different observers, which also makes no sense. Physics was cornered, with no way out.

Ernst Mach, who had a notorious distaste for "fake" paradigms (he believed that Physics had no business talking about things that couldn't be experimented upon),[4] proposed that Physics had created its own dilemma by inventing a nonexistent "æther" in the first place, and we would do well to forget it! He was right, in this case, but it took a less crusty and more optimistic genius to see how such a dismissal could be used to explain all the results at once.

[3] Poor FitzGerald gets less press these days, alas.

[4] Mach would have had apoplexy over today's *quarks* — but that's a story for a later Chapter!

23.4 Einstein's Simple Approach

At this time, Albert Einstein was working as a clerk in the patent office in Zürich, a position which afforded him lots of free time to toy with crazy ideas. Aware of this dilemma, he suggested the following approach to the problem: since we have to give up some part of our common sense, why not simply take both the experiments and MAXWELL'S EQUATIONS at face value and see what the consequences are? No matter how crazy the implications, at least we will be able to remember our starting assumptions without much effort. They are:

- The "Laws of Physics" are the same in one inertial reference frame as in another, regardless of their relative motion.[5]
- All observers will inevitably measure the same velocity of propagation for light in their own reference frame, namely c.

These two postulates are the starting points for Einstein's celebrated SPECIAL THEORY OF RELATIVITY ($\mathcal{STR}$), for which this Chapter is named.[6] The adjective "Special" is there mainly to distinguish the $\mathcal{STR}$ from the *General* Theory of Relativity, which deals with *gravity* and *accelerated reference frames*, to be covered later.

23.5 Simultaneous for Whom?

The first denizen of common sense to fall victim to the $\mathcal{STR}$ was the "obvious" notion that if two physical events occur at the same time in my reference frame, they must occur at the same time in *any* reference frame. This is not true unless they also occur at the same *place*. Let's see why.

[5]An *inertial reference frame* is one that is *not accelerated* — *i.e.* one that is at rest or moving at constant velocity.

[6]It is perhaps unfortunate that the theory was called "Relativity" when in fact it expresses the principle that the "Laws of Physics" are *not* relative; they are the same for *all* reference frames, moving or not! It is the *transformations* between measurements by different observers in relative motion that give weird results. When someone says, "Yeah, Einstein showed that everything is relative," every Physicist within earshot winces. On the other hand, the $\mathcal{STR}$ does explicitly rule out any *absolute* reference frame with respect to which all motion must be measured — thus elevating the negative result of the Michelson-Morley experiment to the status of a First Principle — and does imply that certain phenomena that we always thought *were* absolute, like *simultaneity*, are not! So the name "Relativity" does stimulate appropriate debate.

Einstein was fond of performing imaginary experiments in his head — *Gedankenexperimenten* in German — because the resultant laboratory was larger than anything he could fit into the patent office and better equipped than even today's funding agencies could afford. Unfortunately, the laboratory of the imagination also affords the option of altering the Laws of Physics to suit one's expectations, which means that only a person with a striking penchant for honesty and introspection can work there without producing mostly fantasies. Einstein was such a person, as witnessed by the ironic fact that he used the *Gedankenexperiment* to dismantle much of our common sense and replace it with a stranger truth. Anyway, one of his devices was the laboratory aboard a fast-moving vehicle. He often spoke of *trains*, the most familiar form of transportation in Switzerland to this day; I will translate this into the *glass spaceship* moving past a "stationary" observer [someone has to be designated "at rest," although of course the choice is arbitrary].

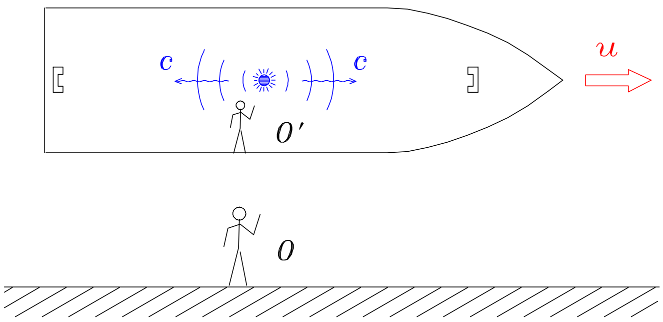

Figure 23.2 A flash bulb is set off in the centre of a glass spaceship (O') at the instant it coincides with a fixed observer O. As the spaceship moves by at velocity u relative to O, the light propagates toward the bow and stern of the ship at the same speed c in both frames.

In Fig. 23.2 both observers (O and O') must measure the same velocity (c) for the light from the flash bulb. The light propagates outward symmetrically in all directions (in particular, to the right and left) from the point where the bulb went off in either frame of reference. In the O' frame, if the two detectors are equidistant from that point they will both detect the light *simultaneously*, but in the O frame the stern of the spaceship moves closer to the source of the flash while the bow moves away, so the stern detector will detect the flash *before* the bow detector!

This is not just an optical illusion or some misinterpretation of the experimental results; this is *actually what happens!* What is *simultaneous* for O' is *not* for O, and *vice versa*. Common sense notwithstanding, SIMULTANEITY is *relative*.

23.6 Time Dilation

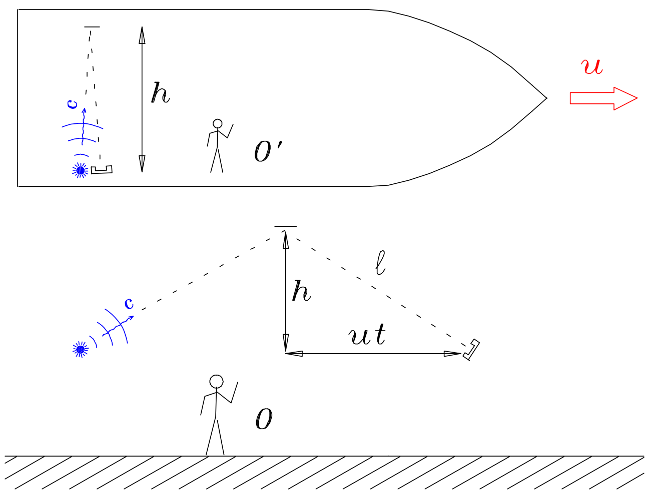

Figure 23.3 A "light clock" is constructed aboard a glass spaceship (reference frame O') as follows: the "tick" of the clock is defined by *one half* the time interval t' required for the light from a strobe light to traverse the width of the ship (a height h), bounce off a mirror and come back, a total distance of $2h$. In the reference frame of a ground-based observer O (with respect to whom the ship is travelling at a velocity u), the light is emitted a distance $2ut$ behind the place where it is detected a time $2t$ later. Since the light has further to go in the O frame (a distance $\ell = \sqrt{h^2 + u^2t^2}$), but it travels at c in both frames, t must be *longer* than t'. This effect is known as TIME DILATION.

Fig. 23.3 pictures a device used by R.P. Feynman, among others, to illustrate the phenomenon of TIME DILATION: a clock aboard a fast-moving vessel (even a normal clock) appears[7] to run *slower* when observed from the "rest frame" — the name we give to the reference frame arbitrarily chosen to be at rest. Now, if we choose to regard the ship's frame as "at rest" (as is the wont of those aboard) and the Earth as "moving," a clock on Earth will appear to be running slowly when observed from the ship! *Who is right?* The correct answer is "both," in utter disregard for common sense. This seems to create a logical paradox, which we will discuss momentarily. But first let's go beyond the qualitative statement, "The clock runs slower," and ask *how much* slower.

For this we need only a little algebra and geometry; nevertheless, the derivation is perilous, so watch carefully.

[7]The term "appears" may suggest some sort of illusion; this is not the case. The clock aboard the spaceship *actually does* run slower in the Earth's rest frame, and *vice versa*.

For O', the time interval described in Fig. 23.3 is simply

$$t' = \frac{h}{c} \qquad \text{so that} \qquad h = ct'$$

whereas for O the time interval is given by

$$t = \frac{\ell}{c} \qquad \text{where} \qquad \ell^2 = h^2 + u^2t^2$$

by the Pythagorean theorem. Expanding the latter equation gives

$$t = \frac{\sqrt{h^2 + u^2t^2}}{c} \qquad \text{or} \qquad c^2t^2 = h^2 + u^2t^2$$

which is not a solution yet because it does not relate t to t'. We need to "plug in" $h^2 = c^2t'^2$ from earlier, to get

$$\begin{aligned} c^2t^2 &= c^2t'^2 + u^2t^2 \\ \text{or} \quad t^2 &= t'^2 + \frac{u^2}{c^2}t^2 \\ \text{or} \quad t^2\left(1 - \beta^2\right) &= t'^2 \end{aligned}$$

where we have recalled the definition $\beta \equiv u/c$. In one last step we obtain

$$t = \frac{t'}{\sqrt{1-\beta^2}} \qquad \text{or} \qquad t = \gamma\, t'$$

where γ is defined as before: $\gamma \equiv 1/\sqrt{1-\beta^2}$.

This derivation is a little crude, but it shows where γ comes from.

23.6.1 The Twin Paradox

Like most "paradoxes," this one isn't. But it sure looks like one at first glance. Suppose two identical twins part company at age twenty; the first twin hops aboard a spaceship of *very* advanced design and heads out for the distant stars, eventually travelling at velocities very close to c, while the second twin stays home at rest. They give each other going-away presents of identical watches guaranteed to keep perfect time under all conditions. At the midpoint of the voyage, while coasting (and therefore in an inertial reference frame), the first twin looks back at Earth with a *very* powerful telescope and observes the second twin's wristwatch. After correcting for some truly illusory effects, he concludes that the first twin's watch is running slower than his and that his twin on Earth must be aging more slowly as well. Meanwhile, the second twin, on Earth, is looking through *his* telescope at the *first* twin's watch (aboard the spaceship) and concludes that the *first* twin is suffering the effects of time dilation and is consequently aging more slowly than *him!* Who is right? Both, at that moment.

Aha! But now we can bring the first twin *home* after his relativistic journey and *compare ages*. Certainly they can't *both* be younger; this truly would create a logical paradox that goes beyond the mere violation of common sense!

What happens? The first twin, who went travelling, is in fact younger now than the twin who stayed home. The paradox is resolved by a meticulous use of the LORENTZ TRANSFORMATIONS, especially if we make use of the graphical gimmick of the LIGHT CONE, to be discussed later.

23.7 Einstein Contraction(?)

We can obtain the concomitant effect of LORENTZ CONTRACTION without too much trouble[8] using the following *Gedankenexperiment*, which is so simple we don't even need a Figure:

Suppose a spaceship gets a nice running start and whips by the Earth at a velocity u on the way to Planet X, a distance x away as measured in the Earth's reference frame, which we call O. [We assume that Planet X is at rest with respect to the Earth, so that there are no complications due to their relative motion.] If the spaceship just "coasts" the rest of the way at velocity u [this is what is meant by an INERTIAL REFERENCE FRAME], then by definition the time required for the voyage is $t = x/u$. But this is the time *as measured in the Earth's reference frame*, and we already know about TIME DILATION, which says that the duration t' of the trip *as measured aboard the ship* (frame O') is *shorter* than t by a factor of $1/\gamma$: $t' = t/\gamma$.

Let's look at the whole trip from the point of view of the observer O' aboard the ship: since our choice of who is at rest and who is moving is perfectly arbitrary, we can choose to consider the *ship* at rest and the Earth (and Planet X) to be hurtling past/toward the ship at velocity u. As measured in the ship's reference frame, the distance from the Earth to Planet X is x' and we must have $u = x'/t'$ by definition. But we also must have $u = x/t$ in the other frame; and by symmetry they are both talking about the same u, so

$$\frac{x'}{t'} = u = \frac{x}{t}$$

and since $t = \gamma t'$ we must also have

$$x = \gamma x'.$$

That is, the distance between fixed points, as measured by the space traveller, is *shorter* than that measured by stay-at-homes on Earth by a factor of $1/\gamma$. This is because the Earth and Planet X represent the *moving* system as measured from the ship. This effect is known as LORENTZ CONTRACTION; it has nothing whatsoever to do with "æther drag!" So one might wonder why it isn't called "Einstein contraction," since we calculated it the way Einstein would have.

Of course, the effect works both ways. The *length of the spaceship*, for instance, will be *shorter* as viewed from the Earth than it is aboard the spaceship itself, because in this case the length in question is *in* the frame that moved *with respect to* the Earth. The sense of the contraction effect can be remembered by this mnemonic:

$$\textit{Moving rulers are shorter.} \qquad (17)$$

However, it is possible to conjure up situations that defy common sense and thus are often (wrongly) described as "paradoxes."

23.7.1 The Polevault Paradox

I have a favourite *Gedankenexperiment* for illustrating the peculiarities of LORENTZ CONTRACTION: picture a *polevaulter* standing beside a 10 foot long barn with a 10 foot polevault pole in her hands. Tape measures are brought out and it is confirmed to everyone's satisfaction that the pole is exactly the same length as the barn. Got the picture? Now the barn door is opened — no tricks

[8] I haven't shown all the false starts in which I got the wrong answer using what seemed like perfectly logical arguments.... Here's a good one:

We can obtain the concomitant effect of LORENTZ CONTRACTION in a sloppy way merely by referring back to Fig. 23.2: let x be the distance between the flash bulb and the forward detector, as measured by the observer O on the ground, and let x' be the same distance as measured by the observer O' aboard the spaceship. Assume that O stretches out a tape measure from the place where the flash bulb is set off (say, by a toggle switch on the outer hull of the spaceship which gets hit by a stick held up by O as O' flies by) to the position of the detector in the O frame at the instant of the flash. That way we don't need to worry about the *position* of the detector in the O frame when the light pulse actually arrives there some time later; we are only comparing the *length* of the spaceship in one frame with the same length in the other. [It may take a few passes of the spaceship to get this right; but hey, this is a *Gedankenexperiment*, where resources are cheap!] Then the time light takes to traverse distance x', according to O', is $t' = x'/c$, whereas the time t for the same process in the rest frame is $t = x/c$. Therefore, if (from TIME DILATION) t is *longer* than t' by a factor γ, then x must also be *longer* than x' by the same factor if both observers are using the same c.

Simple, eh? Unfortunately, I got the wrong answer! Can you figure out why?

— and our intrepid polevaulter walks back a few parsecs to begin her run up.

Suppose we permit a certain amount of fantasy in this *Gedankenexperiment* and imagine that Superwoman, a *very* adept polevaulter, can run with her pole at a velocity $u = 0.6c$. (Thus $\beta = 0.6$ and $\gamma = 1.25$ — check it yourself!) This means that as she runs past a stationary observer her 10 foot pole turns into a 8 foot pole due to LORENTZ CONTRACTION. On the other hand, in her own reference frame she is still carrying a 10 foot pole but the barn is now only 8 feet long. She runs into the barn and the attendant (Superman) slams the barn door behind her.

From Superwoman's point of view, the following sequence of events occurs: first the end of her pole smashes through the end of the barn, and then[9] (somewhat pointlessly, it seems) the barn door slams behind her. A few nanoseconds later she herself hits the end of the barn and the whole schmier explodes in a shower of elementary particles — except for Superwoman and Superman, who are (thankfully) invulnerable.

Superman sees it differently. He has no trouble shutting the barn door behind Superwoman before her polevault pole hits the other end of the barn, so he has successfully performed his assignment — to get Superwoman and her polevaulting skills hidden away inside the barn for the two nanosecond period that the scout for the Olympic Trials happens to be looking this way. What happens after that is pretty much the same as described by Superwoman.

Imagine that you have been called in to mediate the ensuing argument. Who is right? Can you counsel these two Superbeings out of a confrontation that might devastate the surrounding landscape? Or will this become the Parent of all Battles?

Well, if they want to fight they will fight, of course; but the least you can do is point out that objectively there is nothing to fight about: *they are both right!* When you think about it you will see that they have both described the same *events*; it is only the *sequence* of the events that they disagree on. And *the sequence of events is not necessarily the same* for two observers in relative motion! It all comes back to the RELATIVITY OF SIMULTANEITY and related issues. For Superwoman the pole hits the wall before the door slams, while for Superman the door slams before the pole hits the wall. Both events occur for both observers, but the sequence is different.[10]

[9]It takes about 3.4 ns [nanoseconds, 10^{-9} s] to go 2 feet at a velocity of $0.6c$.

[10]If the door were at the *far* end of the barn (where the pole hits), there could be no such disagreement, since two events at the *same place* and the same time are for all intents and purposes part of the *same event*. It is only events *separated in space* about which such differences of opinion can arise.

23.8 Relativistic Travel

Numerous misconceptions have been bred by lazy science fiction ($\mathcal{SF}$) authors anxious to circumvent the limitations imposed by the $\mathcal{STR}$. Let's examine these limitations and ask whether in fact they restrict space-flight options as severely as $\mathcal{SF}$ fans have been led to believe.

The first and most familiar restriction is the familiar statement, "You can't ever go quite as fast as light." Why is this? Well, consider the behaviour of that ubiquitous scaling factor γ as $u \to c$ (*i.e.*, as $\beta \to 1$): as β gets closer and closer to unity, $(1 - \beta)$ gets closer and closer to zero, as does its square root, which means that γ "blows up" (becomes infinite) as $u \to c$. TIME DILATION causes clocks aboard fast-moving spaceships to *freeze* completely and LORENTZ CONTRACTION causes the length of the ship (in the direction of its motion) to *squash* to nothing, if $u \to c$. [As observed by Earth-bound telescopes, of course.] Worse yet, if we *could* achieve a velocity *greater* than c, time would *not* run backwards [or any of the other simplistic extrapolations tossed off in mediocre $\mathcal{SF}$]; rather the time-dilation / Lorentz-contraction factor γ becomes *imaginary* — in other words, *there is no such physical solution* to the LORENTZ TRANSFORMATION equations! At least not for objects with masses that are *real* in the mathematical sense. [I will deal with the hypothetical *tachyons* in a later section.] Another way of understanding why it is impossible to reach the speed of light will be evident when we begin to discuss RELATIVISTIC KINEMATICS in the next Chapter.

So there is no way to get from here to another star 10 light years distant in less than ten years — *as time is measured on Earth!* However, contrary to popular misconceptions, this does *not* eliminate the option of relativistic travel to distant stars, because the so-called "subjective time"[11] aboard the spaceship is far shorter! This is because in the traveller's reference frame the *stars* are moving and the distances between them (in the direction of motion) *shrink* due to LORENTZ CONTRACTION.

It is quite interesting to examine these effects quantitatively for the most comfortable form of relativistic travel: constant acceleration at $1g$ (9.81 m/s^2) as measured in the spaceship's rest frame, allowing shipboard life to con-

[11]Time measured aboard the spaceship is no more "subjective" than time on Earth, of course; this terminology suggests that the experience of the traveller is somehow bogus, which is not the case. Time *actually does* travel more slowly for the moving observer and the distance between origin and destination *actually does* get shorter.

form to the appearance of Earth-normal gravity. I will list two versions of the "range" of such a voyage (measured in the Earth's rest frame) for different "subjective" elapsed times (measured in the ship's rest frame) — one for *arrival at rest* [the only mode of travel that could be useful for "visiting" purposes], in which one must accelerate halfway and then decelerate the rest of the way, and one for a "*flyby*," in which you don't bother to stop for a look [this could only appeal to someone interested in setting a long-distance record].

The practical limit for an *impulse* drive converting mass carried along by ship into a collimated light beam with 100% efficiency is about 10-12 years. Longer acceleration times require use of a "ram scoop" or similar device using *ambient* matter.

Now, what does this say about the real possibilities for relativistic travel? Without postulating any "unPhysical" gimmicks — *e.g.* "warp drives" or other inventions that contradict today's version of the "Laws" of Physics — we can easily compose *SF* stories in which humans (or others) can travel all through our own Galaxy without resorting to suspended animation[12] or other hypothetical future technologies.[13] There is only one catch: As Thomas Wolfe said, *You can't go home again.* Or, more precisely, you can go home but you won't recognize the old place, because all those years it took light to get to your destination and back (that you cleverly dodged by taking advantage of LORENTZ CONTRACTION) still passed normally for the folks back home, now thousands of years dead and gone.

So a wealthy misanthropic adventurer may decide to leave it all behind and go exploring, but no government will ever pay to build a reconnaissance vessel which will not return before the next election. This implies that there may well be visitors from other stars, but they would be special sorts of characters with powerful curiosities and not much interest in socializing. And we can forget about "scouts" from aggressive races bent on colonization, unless they take a *very* long view!

Table 23.1 Distances covered (measured in Earth's rest frame) by a spaceship accelerating at a constant $1g$ (9.81 $\mathrm{m/s^2}$) in its own rest frame.

Elapsed Time	**Distance Travelled** (Light Years)	
aboard ship (years)	**Arriving at Rest**	**"Fly-by"**
1	0.063	0.128
2	0.98	2.76
3	2.70	9.07
4	5.52	26.3
5	10.26	73.2
6	18.14	200.7
7	31.14	547.3
8	52.6	1,490
9	88	4,050
10	146	11,012
11	244	29,936
12	402	81,376
13	665	221,200
14	1,096	601,300
15	1,808	1,635,000
16	2,981	4,443,000
17	4,915	12,077,000
18	8,103	32,830,000
19	13,360	89,241,000
20	22,000	243,000,000
21	36,300	659,000,000
22	59,900	1,792,000,000
23	99,000	4,870,000,000
24	163,000	13,200,000,000
25	268,000	36,000,000,000
26	442,000	98,000,000,000
27	729,000	(present diam.
28	1,200,000	of universe
29		thought to be
30		less than about
		30,000,000,000)

[12] The idea of suspended animation is a good one and I find it plausible that we may one day learn to use it safely; but it does not quite fall into the category of a simple extrapolation from known technology — yet.

[13] Except for the "ramscoop" technology and the requisite shields against the thin wisp of ambient matter (protons, electrons,...) inhabiting interstellar space, which is converted into high-energy radiation by virtue of our ship's relative motion. Minor details.

23.9 Natural Units

As I mentioned in the Chapter on UNITS AND DIMENSIONS, in any context where the *speed of travel* is virtually (or, in this case, exactly) a *constant*, people automatically begin to express *distances* in *time* units. [Q: "How far is is from New York to Boston?" A: "Oh, about three hours."] This is equivalent to defining the *speed of travel* to be a dimensionless constant of magnitude 1. Relativistic Physics is no different. Anyone who has to discuss relativistic phenomena at any length will usually slip into "NATURAL UNITS" where

$$c = 1$$

and distance and time are measured in the same units. You get to pick your favourite unit — seconds, meters, light years or (as we shall see later) inverse masses! The list is endless. Then β is just "the velocity" measured in natural units and the calculations become much simpler. But you have to convert all your other units accordingly, and this can be interesting. It does take a little getting used to, but the exercise is illuminating.

23.10 A Rotational Analogy

If we compare the LORENTZ TRANSFORMATIONS with the GALILEAN TRANSFORMATIONS, several striking qualitative features are apparent: the first is the multiplicative factor γ which describes both TIME DILATION and LORENTZ CONTRACTION; the second is the fact that *time* and *space* get *mixed together* by the LORENTZ TRANSFORMATION — a blasphemy in the paradigm of classical Physics.

The latter weirdness is going to be confusing no matter what we do; is there any way to at least make it *look* familiar? What we need is an *analogy* with something that *does* "make sense" and is still intact. Fortunately there is a precedent for a transformation that *mixes coordinates*, namely the ROTATION.

23.10.1 Rotation in Two Dimensions

Suppose we have a point A in a plane with perpendicular x and y coordinate axes scribed on it, as pictured in Fig. 23.4.

We can scribe a *different* pair of perpendicular coordinate axes x' and y' on the same plane surface using dashed lines by simply *rotating* the original coordinate axes by an angle θ about their common origin, the coordinates of which are $(0, 0)$ in *either* coordinate system.

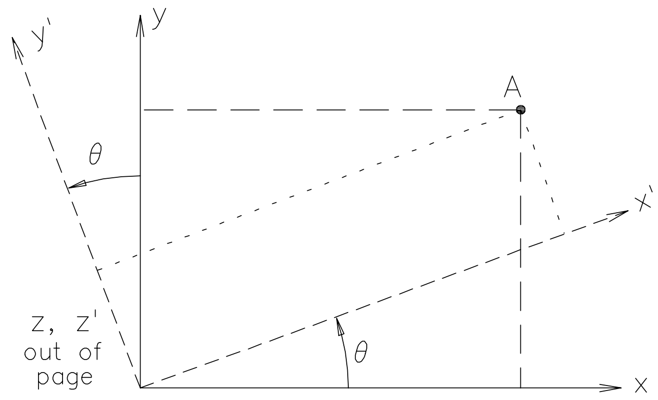

Figure 23.4 A fixed point A can be located in a plane using either of two coordinate systems O (x, y) and O' (x', y') that differ from each other by a rotation of θ about the common origin $(0, 0)$.

Now suppose that we have the coordinates (x_A, y_A) of point A in the original coordinate system and we would like to *transform* these coordinates into the coordinates (x'_A, y'_A) of the *same point* in the new coordinate system.[14] How do we do it? By trigonometry, of course. You can figure this out for yourself. The transformation is

$$x' = x\cos(\theta) + y\sin(\theta) \qquad (18)$$
$$y' = -x\sin(\theta) + y\cos(\theta) \qquad (19)$$

23.10.2 Rotating Space into Time

If we now look at just the x and t part of the LORENTZ TRANSFORMATION [leaving out the y and z parts, which don't do much anyway], we have

$$x' = \gamma x - \gamma\beta\, ct \qquad (20)$$
$$ct' = -\gamma\beta\, x + \gamma\, ct \qquad (21)$$

— *i.e.*, the LORENTZ TRANSFORMATION "sort of" *rotates* the *space* and *time* axes in "sort of" the same way as a normal rotation of x and y. I have used ct as the time axis to keep the units explicitly the same; if we use "natural units" ($c = 1$) then we can just drop c out of the equations completely and the analogy becomes obvious. However, you should resist the temptation to think

[14] This situation might arise if an architect suddenly discovered that his new plaza had been drawn from coordinates laid out by a surveyor who had aligned his transit to magnetic North while standing next to a large industrial electromagnet. The measurements are all OK but they have to be converted to true latitude and longitude!

of the LORENTZ TRANSFORMATION as "just a rotation of space and time into each other." If we "boost" the O' frame by some large relative velocity in the *negative* x direction and try to plot up x' and ct' on the same graph as (x, ct) then we get a weird picture.

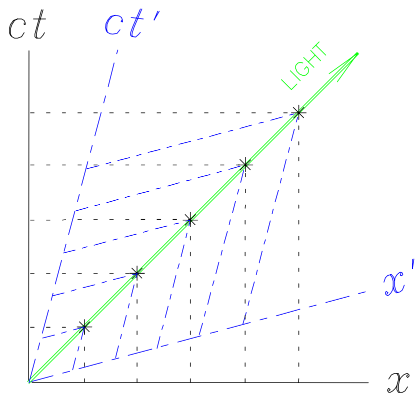

Figure 23.5 An attempt to draw (x', ct') coordinates on the same graph as the (x, ct) coordinates. The result is misleading because the spatial surface on which it is drawn obeys EUCLIDEAN geometry (the invariant length of an interval is the square root of the sum of the squares of its two perpendicular components) whereas spacetime obeys the MINKOWSKI metric: the invariant "length" of a spacetime interval (the PROPER TIME) is equal to $c^2t^2 - x^2$, not $c^2t^2 + x^2$). You may *think* of the LORENTZ TRANSFORMATION as a sort of rotation, but you can't *draw* it as a rotation, because you don't have Minkowski paper!

Proper Time and Lorentz Invariants

The most important important difference between ordinary ROTATIONS and the LORENTZ TRANSFORMATIONS is that the former preserve the RADIUS distance

$$r = \sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2} \qquad (22)$$

of point A from the origin, whereas the latter preserve the PROPER TIME τ of an event:

$$c\tau = \sqrt{c^2t^2 - x^2} = \sqrt{c^2t'^2 - x'^2} \qquad (23)$$

The $-$ sign in the latter is important!

In general, any quantity which we can define (like τ) that will have *the same value* in every inertial reference frame, regardless of relative motion, may be expected to become very precious to our bruised sensibilities. The $\mathcal{STR}$ has dismantled most of our common sense about which physical observables are reliable, universal constants and which depend upon the reference frame of the observer; if we can specifically identify those properties of a quantity that will guarantee its *invariance* under LORENTZ TRANSFORMATIONS, then we can at least count on such quantities to remain reliably and directly comparable for different observers. Such quantities are known as LORENTZ INVARIANTS.

The criterion for LORENTZ INVARIANCE is that the quantity in question be the *scalar product of two 4-vectors*, or any combination of such scalar products. What do we mean by *4-vectors*? {Space and time} make the classic example, but we can *define* a *4-vector* to be *any 4-component quantity that transforms like spacetime.* That is, $a_\mu = \{a_0, a_1, a_2, a_3\}$ — where a_0 is the "timelike" component (like ct) and $\{a_1, a_2, a_3\}$ are the three "spacelike" components (like x, y, z) — is a *4-vector* if a "boost" of u in the x direction gives

$$\begin{aligned} a'_0 &= \gamma(a_0 - \beta a_1) \\ a'_1 &= \gamma(a_1 - \beta a_0) \\ a'_2 &= a_2 \\ a'_3 &= a_3 \end{aligned}$$

just like for $x_\mu = \{ct, x, y, z\}$. The most important example (other than x_μ itself) is $p_\mu = \{E, p_x, p_y, p_z\}$, the ENERGY-MOMENTUM 4-vector, which we will encounter next.