Vectors

August 4, 1998

A Generalization of "Orthogonality"

The definition of a *vector* as an entity with both magnitude and direction can be generalized if we realize that "direction" can be defined in more dimensions than the usual 3 spatial directions, "up-down, left-right, and back-forth," or even other dimensions *excluding* these three. The more general definition would read,

> *DEFINITION:* a **vector** quantity is one which has *several independent attributes* which are *all measured in the same units* so that "transformations" are possible. (This last feature is only essential when we want the advantages of mathematical manipulation; it is not necessary for the concept of multi-dimensional entities.)

We can best illustrate this generalization with an example of a vector that has nothing to do with 3-D space:

> EXAMPLE: the **Cost of Living**, $\vec{\mathbf{C}}$, is in a sense a true vector quantity (although the Cost of Living *index* may be properly thought of as a *scalar*, as we can show later).

To construct a simple version, the Cost of Living can be taken to include:

- C_1 = **housing** (e.g., monthly rent);
- C_2 = **food** (e.g., cost of a quart of milk);
- C_3 = **medical** service (e.g., cost of a bottle of aspirin);
- C_4 = **entertainment** (e.g., cost of a movie ticket);
- C_5 = **transportation** (e.g., bus fare);
- $C_6 \ldots C_7 \ldots$ etc. (a finite number of "*components*.")

Thus we can write $\vec{\mathbf{C}}$ as an ordered sequence of numbers representing the values of its respective "components":

$$\vec{\mathbf{C}} = (C_1, C_2, C_3, C_4, C_5, \ldots) \tag{1}$$

We would normally go on until we had a reasonably "complete" list – i.e., one with which the cost of any additional item we might imagine could be expressed *in terms of* the ones we have already defined. The technical mathematical term for this condition is that we have a "*complete basis set*" of components of the Cost of Living.

Now, we can immediately see an "inefficiency" in the way $\vec{\mathbf{C}}$ can been "composed:" As recently as 1975, it was estimated to take approximately one pound of gasoline to grow one pound of food in the U.S.A.; therefore the cost of **food** and the cost of **transportation** are obviously *not independent*! Both are closely tied to the cost of **oil**. In fact, a large number of the components of the cost of living we observe are intimately connected to the cost of oil (among other things). On the other hand (before we jump to the fashionable conclusion that these two components should be replaced by oil prices alone), there is *some* measure of independence in the two components. How do we deal with this quantitatively?

To reiterate the question more formally, how do we quantitatively describe the extent to which certain components of a vector are superfluous (in the sense that they merely represent combinations of the other components) *vs.* the extent to which they are truly "independent?" To answer, it is convenient to revert to our old standby, the (graphable) analogy of the **distance** vector in *two dimensions*.

Suppose we wanted to describe the position of any point P in the "$x - y$ plane." We could draw the two axes "a" and "b" shown above. The position of an arbitrary point P is uniquely determined by its (a, b) coordinates, defined by the prescription that to change a we move parallel to the a-axis and to change b we move parallel to the b-axis. This is a unique and quite legitimate way of specifying the position of any point (in fact it is often used in crystallography where the orientation of certain crystal axes is determined by nature); yet there is something vaguely troubling about this choice of coordinate axes. What is it? Well, we have an intuitive sense of "up-down" and "sideways" as being *perpendicular*, so that if something moves "up" (as we normally think of it), in the above description the values of both a and b will change. But isn't our intuition just the result of a well-entrenched convention? If we got used to thinking of "up" as being in the "b" direction shown, wouldn't this cognitive dissonance dissolve?

No. In the first place, nature provides us with an unambiguous characterization of "down:" it is the direction in which things fall when released; the direction a string points when tied to a plumb bob. "Sideways," similarly, is the direction defined by the surface of an undisturbed liquid (as long as we neglect the curvature of the Earth's surface). That is, gravity fixes our notions of "appropriate" geometry. But is this in turn arbitrary (on nature's part) or is there some good reason why "independent" components of a vector should be perpendicular? And what exactly do we *mean* by "perpendicular," anyway? Can we define the concept in a way which might allow us to generalize it to other kinds of vectors besides space vectors?

The answer is bound up in the way Euclid found to express the geometrical properties of the world we live in; in particular, the "metric" of space – the way we define the **magnitude** (*length*) of a vector. Suppose you take a ruler and turn it at many angles; your idea of the *length* of the ruler is *independent* of its *orientation*, right? Suppose you used the ruler to make off distances along two perpendicular axes, stating that these were the horizontal and vertical components (x, y) of a distance vector. Then you use the usual "parallelogram rule" to locate the tip of the vector, draw in a line from the origin to that point, and put an arrowhead on the line to indicate that you have a vector. Call it "$\vec{\mathbf{r}}$". *You can use the same ruler, held at an angle, to measure the length r of the vector.* Pythagoras gave us a formula for this length. It is

$$r = \sqrt{x^2 + y^2}. \tag{2}$$

This formula is the key to Euclidean geometry, and is the working definition of perpendicular axes: x and y are perpendicular if and only if Eq. (2) holds. It does *not* hold for "a" and "b" described earlier!

You may feel that this "metric" is obvious and necessary from first principles; it is not. If you treat this formula as correct using the

Earth's surface as the "$x - y$ plane" you will get good results until you start measuring off distances in the thousands of miles; then you will be 'way off! Imagine for instance the perpendicular lines formed by two longitudes at the North Pole: these same "perpendicular" lines *cross again* at the South Pole!

Well, of course, you say; that is because the Earth's surface is *not* a plane; it is a sphere; it is *curved*. If we didn't feign ignorance of that fact, if we did our calculations in *three* dimensions, we would always get the right answers. Unfortunately not. The space we live in is actually *four*-dimensional, and it is *not* flat, *not* "Euclidean," in the neighborhood of large masses. Einstein helped open our eyes to this fact, and now we are stuck with a much more cognitively complex understanding.

But we have to start somewhere, and the space we live in from day to day in "pretty Euclidean," and it is only in the violation of sensible approximations that modern physics is astounding, so we will pretend that only Euclidean vector spaces are important. (Do you suppose there is a way to generalize our definition of "perpendicularity" to include non-Euclidean space as well?)

Finally returning to our original example, we would like to have $\vec{\mathbf{C}}$ expressed in an "orthogonal, complete basis", $\vec{\mathbf{C}} = (C_1, C_2, C_3, C_4, C_5, \ldots)$, so that we can define the **magnitude** of $\vec{\mathbf{C}}$ by

$$C = \mid \vec{\mathbf{C}} \mid = \sqrt{C_1^2 + C_2^2 + C_3^2 + \ldots} \tag{3}$$

("Orthogonal" and "normal" are just synonyms for "perpendicular.") We could call $\vec{\mathbf{C}}$ the "Cost of Living Index" if we liked. There is a problem now. Our intuitive notion of "independent" components is tied up with the idea that one component can change without affecting another; yet as soon as we attempt to be specific about it, we find that we cannot even define a criterion for formal and exact independence (orthogonality) without generating a new notion: the idea of a magnitude as defined by Eq. (3). Does this definition agree with out intuition, the way the "ruler" analogy did? Most probably we *have* no intuition about the "magnitude" of the "cost of living vector." So we have created a new concept – not an arbitrary concept, but one which is guaranteed to have a large number of "neat" consequences, one we will be able to do calculations with, make transformations of, and so on. In short, a "rich" concept.

There is another problem, though; while we can easily test our space vectors with a ruler, there is no unambiguous "ruler" for the "cost of living index." Furthermore, we may make the *approximation* that the cost of tea bags is orthogonal to the cost of computer maintenance, but in so "messy" a business as economics we will never be able to prove this rigorously. There are too many "hidden variables" influencing the results in ways we do not suspect. This is too bad, but we can still live with the imperfections of an approximate model if it serves us well.