Experiment 5

Electric Fields

Introduction

The integral form of Gauss's Law relates the integral of the electric field over some surface to the charge enclosed by that surface.

$$\int \epsilon_0 \mathbf{E} \cdot d\mathbf{A} = Q_{\text{encl}} = \int \rho dV \tag{1}$$

where ρ is the charge density and the volume integral extends over the volume enclosed by the surface in the first integral. Gauss's Law can also be written in its differential form:

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{2}$$

The electric **potential** ϕ is related to the electric field:

$$\mathbf{E} = -\nabla\phi \tag{3}$$

We can then substitute Eqn. 3 into Eqn. 2

$$\begin{aligned} \epsilon_0 \nabla \cdot (-\nabla\phi) &= \rho & (4) \\ \epsilon_0 \nabla^2 \phi &= -\rho & (5) \end{aligned}$$

Equation 5 is known as Poisson's Equation. If there is no free charge, then $\rho = 0$ and Eqn. 5 reduces to Laplace's Eqn.:

$$\nabla^2 \phi = 0 \tag{6}$$

In Cartesian coordinates, Laplace's Eqn. reads:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{7}$$

Solving Laplace's Equation on a Discrete Lattice

We can expand the potential function $\phi(x,y,z)$ in a Taylor series around some point in space (x_0,y_0,z_0) to give the potential at the six points $(x_0 \pm \delta, y_0, z_0)$, $(x_0, y_0 \pm \delta, z_0)$ and $(x_0, y_0, z_0 \pm \delta)$ which symmetrically surround (x_0,y_0,z_0).

To third order in δ (*i.e.* δ^3):

$$\begin{aligned}
\phi(x_0 \pm \delta, y_0, z_0) &\approx \phi(x_0,y_0,z_0) \pm \delta\frac{\partial\phi}{\partial x} + \frac{1}{2}\delta^2\frac{\partial^2\phi}{\partial x^2} \pm \frac{1}{6}\delta^3\frac{\partial^3\phi}{\partial x^3} && (8)\\
\phi(x_0, y_0 \pm \delta, z_0) &\approx \phi(x_0,y_0,z_0) \pm \delta\frac{\partial\phi}{\partial y} + \frac{1}{2}\delta^2\frac{\partial^2\phi}{\partial y^2} \pm \frac{1}{6}\delta^3\frac{\partial^3\phi}{\partial y^3} && (9)\\
\phi(x_0, y_0, z_0 \pm \delta) &\approx \phi(x_0,y_0,z_0) \pm \delta\frac{\partial\phi}{\partial z} + \frac{1}{2}\delta^2\frac{\partial^2\phi}{\partial z^2} \pm \frac{1}{6}\delta^3\frac{\partial^3\phi}{\partial z^3} && (10)
\end{aligned}$$

The average of those six values of the potential at points surrounding (x_0,y_0,z_0) is then, to third order in δ:

$$\phi(x_0,y_0,z_0) + \frac{\delta^2}{6}\left[\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right] \qquad (11)$$

If the potential satisfies Laplace's equation ($\nabla^2\phi = 0$), then the average of the six neighbouring values of the potential is just the potential at the point in question (where we have neglected terms in δ^4 and higher).

We can now use this result to calculate the potential $\phi(x,y,z)$ numerically for a given set of boundary conditions. We rely on the fact (demonstrated above) that the value of a harmonic function at some point in space is equal to its average over the neighbourhood of the point. We represent the potential function ϕ by values at an array of discrete points, including discrete points at the boundaries. We then adjust the values at non-boundary points until each value is equal to the average of its neighbouring values. To do this, we employ an approximate technique: the **relaxation method**.

We start with the boundary values fixed at their given values and assign arbitrary values to the interior points. Next, we cycle through each of the interior points and assign its value to the average of its neighbouring 4 (in two dimensions) adjacent grid points. We repeat this over and over again until the changes in the values become acceptably small. We could do this calculation by hand, though we would quickly find the iterations quite tiresome. Fortunately, we can use a spreadsheet program to quickly do these calculations for us.

Measuring the Electric Potential

There is no simple and sufficiently sensitive device which permits the measurement of the electric field intensity at a point in the free space surrounding a system of electrically charged conductors. Instead, we will study fields in two dimensions, on a sheet of high resistance slightly-conducting paper. The experimental setup is shown in the figure below. The source of the charges is a 60 Hz oscillator (function generator). Wires from the oscillator are attached to electrodes having

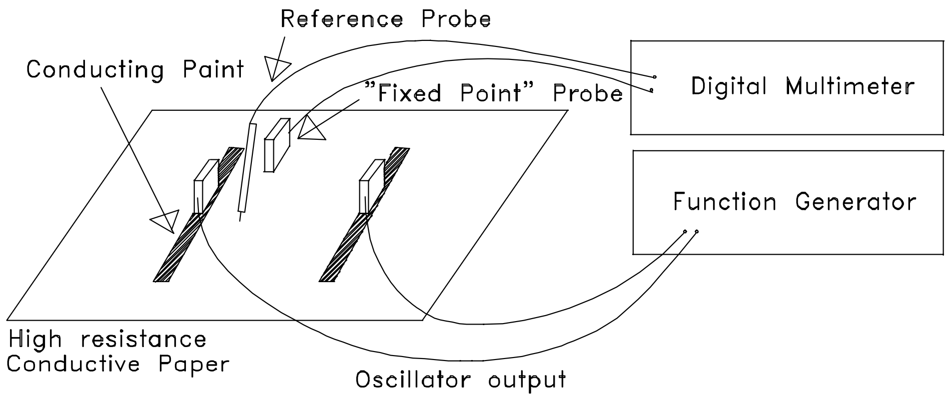

Figure 5.1 Apparatus for measuring electric potential in two dimensions.

protruding pins, which are then connected directly into the paper or into large conductors which have been painted on the paper. Minute electric currents are established in the paper along the electric field lines between the point charges or painted conductors. The field can then be studied using two probes connected to a digital voltmeter (DVM). To do this, the probes are placed at two points on the paper, with the voltmeter then indicating the potential difference between the two points. When the voltmeter reads 0 Volts, there is no potential difference between the probes and therefore the probes lie at two points on the same equipotential "surface" (in two dimensions, an equipotential curve). Since the electric field is everywhere orthogonal to the equipotential curves, the field lines can be mapped once several equipotential curves have been delineated.

To plot each equipotential curve, a "fixed point" probe is inserted at some point, while a reference probe is moved to locate the points at which a null potential difference is obtained. The locus of these equipotential points should be marked on the conducting paper. To plot a new equipotential curve, the fixed point probe is moved to another point and the process is repeated.

The experimental arrangement imposes a distortion in plotting the field distribution. Since currents are confined to the paper itself, the current (and therefore the field) at the edge of the paper is always parallel to the edge. Hence distortion of the field near the edge of the paper must be expected. Whatever the directions of the equipotential surfaces might be in an unlimited medium, they will always be found perpendicular to the edge of the paper and therefore distorted into this direction over a considerable distance away from the edge.

The various shaped conductors are painted onto the coated paper using highly conductive paint. Note that to be effective this paint must be applied in a uniform layer and allowed to dry for at least ten minutes.

You will study the field of two point charges of equal magnitude–both in the case that the point charges are of opposite sign and in the case that they are of the same sign–and you will study the field between two parallel conducting plates. You will also verify the following theorem:

Electrostatic Shielding Theorem:

If a hollow conductor of arbitrary shape is placed in an electric field and if there are no charges in the cavity of the conductor, then the field in the cavity is zero. It follows that the interior of the cavity is an equipotential region, *i.e.*, all points in the cavity are at the same potential as the conductor itself.

Finally, you will make quantitative measurements of the field's dependence on distance in the region between two parallel conducting plates.

Measuring the Potential

Turn on the DVM and the function generator. Set the DVM to the 200 mV AC scale by selecting AC, Volts, and 200m on the volts scale. Set the function generator for a 60 Hz sine wave by selecting the range at 100, setting the frequency dial to .6 and selecting the sine wave function. The function generator should also be tuned to a 150 mV output. To do this, first set the volts-out to 0-2 V. Then connect the function generator output to the voltmeter, and adjust the amplitude dial on the function generator until the voltmeter measures 150 mV. You can now disconnect the function generator from the voltmeter, since this setting will not need to be changed during the lab.

Connect the leads from the function generator to the point contacts and insert them at appropriate locations on the conducting paper. Connect the reference probe and the fixed point probe to the voltmeter.

Calculating the Potential

Start up the PC and start `WINDOWS` (by entering `win` at the `C:` prompt). We will use the EXCEL spreadsheet program for calculating the potential using the relaxation method described above. We rely on the fact the potential at any interior point in a system will be equal to the average of its neighbours.

In a new workbook, define the region of cells that will correspond to your system. Choose a coordinate system, such as assuming that distance between adjacent cells is 2mm. You may wish to set the column widths to be fairly narrow (say 3 digits) so that more cells will be visible on the screen at a time.

To begin, set the value of each cell in the part of the spreadsheet corresponding to the system begin studied to zero. You can use the copy and paste functions to do this easily. We will leave the potential at the boundaries of our system at zero. Pick an interior cell. Set its value to be the average of its neighbours. For example, if you picked cell `D3`, you would set its value to `=average(D2,E3,D4,C3)`. You can then copy this value to all of the interior cells in the spreadsheet. You may get a message that `EXCEL` can't resolve circular references, but that is okay.

We are now ready to change the potentials of certain cells to fixed values. This corresponds to putting fixed potentials on our electrodes in the Measurement part of the experiment. Start with a point at a fixed potential. Choose a point somewhere near the center of your system and change its value to a fixed value (say 100). At this point, you are ready to perform the relaxation calculation. From the `TOOLS` item on the Toolbar, select `OPTIONS`, then the `Calculation` window. There is a checkbox for Iteration, as well as a place to enter the maximum number of iterations and the size of the change to stop at for the calculated values. The default values are probably fine. Check the box, then watch the system perform the relaxation calculation.

You can use `EXCEL` to graph the resulting potential as a 3-D surface plot. To do this, select the region of the spreadsheet you wish to plot with the mouse, then select the `Chart` symbol. Select an empty region on your spreadsheet for the graph, then answer the questions for the Legend and axis labels etc.

Procedure

5.1 Part 1

Following the instructions for measuring the potential, plot a series of equipotential curves (staying away from the edges of the paper if possible) and draw a full pattern of field lines for the following configurations of charged conductors:

A. Two equal but opposite sign "point" charges (with each oscillator lead connected to a pin on the bare paper).

B. Two equal and same sign "point" charges (with one oscillator lead connected in common to two separated pins while the other lead is connected to several pins at the edges of the paper, i.e., "at infinity").

C. Two parallel straight conductors having opposite sign charges (with a separate oscillator lead connected to each). Leave at least six inches between the conductors.

D. Some other arrangement of charged conductors which you feel is interesting.

5.2 Part 2

Verify the "Electrostatic Shielding" theorem described previously. You can test this theorem by painting a closed conducting loop (e.g., a circle) in a region of known electric field and investigating the potential inside this loop. You should use the sheet of conducting paper previously used in Part 1(C) and paint a closed loop in the region between the two parallel straight conductors.

5.3 Part 3

In addition to the null method for plotting equipotentials, the digital voltmeter can be used to make direct quantitative measurements of the potential difference between two points. Recall that for this lab you have set the total potential difference between the electrodes at 150 mV.

Using one of the long strips of conducting paper available in the lab, paint a conducting bar at each end and insert one electrode into each bar. Place one DVM probe on one of the bars, and with the other DVM probe measure the potential difference at some other point on the strip. Verify that the potential varies linearly with position by measuring the potential difference for various distances along the strip. Does this indicate a uniform electric field?

5.4 Part 4

Using the spreadsheet, solve Laplace's equation numerically for each of the electrode configurations you've measured. Have `EXCEL` graph the potential surfaces in each case. Compare your calculated and measured results.